FIXED POINTS IN NON-INVARIANT PLANE CONTINUA

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ABSTRACT. If $f : [a, b] \to \mathbb{R}$, with $a < b$, is continuous and such that a and b are mapped in opposite directions by f, then f has a fixed point in I. Suppose that $f : \mathbb{C} \to \mathbb{C}$ is map and X is a continuum. We extend the above for certain continuous maps of dendrites $X \to D, X \subset D$ and for positively oriented maps $f : X \to \mathbb{C}, X \subset \mathbb{C}$ with the continuum X not necessarily invariant. Then we show that in certain cases a holomorphic map $f : \mathbb{C} \to \mathbb{C}$ must have a fixed point a in a continuum X so that either $a \in \text{Int}(X)$ or f exhibits rotation at a.

1. INTRODUCTION

By $\mathbb{C}$ we denote the plane and by $\mathbb{C}^\infty$ the Riemann sphere. The fixed point problem, attributed to [Ste35], is one of the central problems in plane topology. It can be formulated as follows.

Problem 1.1. *Suppose that $f : \mathbb{C} \to \mathbb{C}$ is continuous and $f(X) \subset X$ for a non-separating continuum X. Does there always exist a fixed point in X?*

As is, Problem 1.1 is not yet solved. The most well-known particular case for which it is solved is that of a map of a closed interval $I = [a, b]$, $a < b$ into itself in which case there must exist a fixed point in I. In fact, in this case a more general result can be proven of which the existence of a fixed point in an invariant interval is a consequence.

Namely, instead of considering a map $f : I \to I$ consider a map $f : I \to \mathbb{R}$ such that either (a) $f(a) \geq a$ and $f(b) \leq b$, or (b) $f(a) \leq a$ and $f(b) \geq b$. Then still there must exist a fixed point in I which is an easy corollary of the Intermediate Value Theorem applied to the function $f(x) - x$. Observe that in this case I need not be invariant under f. Observe also that without the assumptions on the endpoints, the conclusion on the existence of a fixed point inside I cannot be made because, e.g., a shift map on I does not have fixed points at all. The conditions (a) and (b) above can be thought of as boundary conditions imposing restrictions on where f maps the boundary points of I in $\mathbb{R}$.

Our main aim in this paper is to consider some other cases for which Problem 1.1 is solved in affirmative (i.e., the existence of a fixed point in an invariant continuum is established) and replace for them the invariantness of the continuum by boundary

conditions in the spirit of the above "interval version" of Problem 1.1. More precisely, instead of assuming that $f(X) \subset X$ we will make some assumptions on the way that f maps points of $\overline{f(X) \setminus X} \cap X$.

First though let us discuss particular cases for which Problem 1.1 is solved. They can be divided into two categories: either X has specific properties, or f has specific properties. The most direct extension of the "interval particular case" of Problem 1.1 is, perhaps, the following well known theorem (see for example[Nad92]).

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Theorem 1.2. *If $f : D \to D$ is a continuous map of a dendrite into itself then it has a fixed point.*

Here f is just a continuous map but the continuum D is very nice. Theorem 1.2 can be generalized to the case when f maps D into a dendrite $X \supset D$ and certain conditions onto the behavior of the points of the boundary of D in X are fulfilled. This presents a "non-invariant" version of Problem 1.1 for dendrites and can be done in the spirit of the interval case described above. Moreover, with some additional conditions it has consequences related to the number of periodic points of f. The details and exact statements can be found to Section 2 devoted to the dendrites.

Another direction is to consider specific maps of the plane on arbitrary non-separating continua. Let us go over known results here. Cartwright and Littlewood [CL51] have solved Problem 1.1 in affirmative for orientation preserving homeomorphisms of the plane. This result was generalized to all homeomorphisms by Bell [Bel78]. The existence of fixed points for orientation preserving homeomorphisms of the *entire plane* under various conditions was considered in [Bro12, Bro84, Fat87, Fra92, Gui94, FMOT07], and of a point of period two for orientation reversing homeomorphisms in [Bon04].

The result by Cartwright and Littlewood deals with the case when X is an invariant continuum. In parallel with the interval case, we want to extend this to a larger class of maps of the plane (i.e., not necessarily one-to-one) such that certain "boundary" conditions are satisfied. Our tools are mainly based on [FMOT07] and apply to *positively oriented maps* which generalize the notion of an orientation preserving homeomorphism (see the precise definitions in Section 2). Our main topological results are Theorems 3.12 and 3.19. The precise conditions in them are somewhat technical - after all, we need to describe "boundary conditions" of maps of arbitrary non-separating continua. However it turns out that these conditions are satisfied by holomorphic maps (in particular, polynomials), allowing us to obtain a few corollaries in this case, essentially all dealing with the existence of periodic points in certain parts of the Julia set of a polynomial and degeneracy of certain impressions.

2. Preliminaries

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A map $f : X \to Y$ is *perfect* if for each compact set $K \subset Y$, $f^{-1}(K)$ is also compact. *All maps considered in this paper are prefect.* Given a continuum $K \subset \mathbb{C}$, denote by $U_\infty(K)$ the component of $\mathbb{C} \setminus K$ containing infinity, and by $T(K)$ the *topological hull of K*, i.e. the set $\mathbb{C} \setminus U_\infty(K)$. By $\mathbb{S}^1$ we denote the unit circle which we identify with $\mathbb{R} \mod \mathbb{Z}$.

In this section we will introduce a new class of maps which are the proper generalization of an orientation preserving homeomorphism. For completeness we will recall some related results from [FMOT07] where these maps were first introduced.

Definition 2.1 (Degree of a map). Let $f : U \to \mathbb{C}$ be a map from a simply connected domain U into the plane. Let S be a positively oriented simple closed curve in U, and $w \notin f(S)$ be a point. Define $f_w : S \to \mathbb{S}^1$ by

$$f_p(x) = \frac{f(x) - w}{|f(x) - w|}.$$

Then f_w has a well-defined *degree* (also known as the *winding number* of $f|_S$ about w), denoted $\mathrm{degree}(f_w) = \mathrm{win}(f, S, w)$.

Definition 2.2. A map $f : U \to \mathbb{C}$ from a simply connected domain U is *positively oriented* (respectively, *negatively oriented*) provided for each simple closed curve S in U and each point $w \in f(T(S)) \setminus f(S)$, $\mathrm{degree}(f_w) > 0$ ($\mathrm{degree}(f_w) < 0$, respectively).

A holomorphic map $f : \mathbb{C} \to \mathbb{C}$ is a prototype of a positively oriented map. Hence the results obtained in this paper apply to them. However, in general positively oriented maps do not have to be light (i.e., a positively oriented map can map a subcontinuum of $\mathbb{C}$ to a point). Observe also, that for points $w \notin T(f(S))$ we have $\mathrm{degree}(f_w) = 0$.

A map $f : \mathbb{C} \to \mathbb{C}$ is *oriented* provided for each simple closed curve S and each $x \in T(S)$, $f(x) \in T(f(S))$. Every positively or negatively oriented map is oriented (indeed, otherwise there exists $x \in T(S)$ with $f(x) \notin T(f(S))$ which implies that $\mathrm{win}(f, S, f(x)) = 0$, a contradiction). A map $f : \mathbb{C} \to \mathbb{C}$ is *confluent* provided for each subcontinuum $K \subset \mathbb{C}$ and every component C of $f^{-1}(K)$, $f(C) = K$. It is well known that both open and monotone maps (and hence compositions of such maps) of continua are confluent. It follows from a result of Lelek and Read [LR74] that each confluent mapping of the plane is the composition of a monotone map and a light open map. Theorem 2.3 is obtained in [FMOT07].

orient

Theorem 2.3. *Suppose that $f : \mathbb{C} \to \mathbb{C}$ is a surjective map. Then the following are equivalent:*

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(1) *f is either positively or negatively oriented;*
(2) *f is oriented;*
(3) *f is confluent.*

Moreover, if f satisfies these properties then for any non-separating continuum X we have $f(Bd(X)) \supset Bd(f(X))$.

Let X be an non-separating plane continuum. A *crosscut* of $U = \mathbb{C} \setminus X$ is an open arc $A \subset U$ such that $\mathrm{Cl}(A)$ is an arc with exactly two endpoints in $\mathrm{Bd}(U)$. Evidently, a crosscut of U separates U into two disjoint domains, exactly one unbounded.

Let S be a simple closed curve in $\mathbb{C}$ and suppose $g : S \to \mathbb{C}$ has no fixed points on S. Since g has no fixed points on S, the point $z - g(z)$ is never 0. Hence the unit vector $v(z) = \frac{g(z) - z}{|g(z) - z|}$ always exists. Let $z(t)$ be a convenient counterclockwise parameterization of S by $t \in \mathbb{S}^1$ and define the map $\overline{v} = v \circ z : \mathbb{S}^1 \to \mathbb{S}^1$ by

$$\overline{v}(t) = v(z(t)) = \frac{g(z(t)) - z(t)}{|g(z(t)) - z(t)|}.$$

Then $\mathrm{Ind}(\mathrm{g}, \mathrm{S})$, the *index of g on S*, is the *degree* of $\overline{v}$. The following theorem (see, e.g., [FMOT07]) is a major tool in finding fixed points of continuous maps of the plane.

basic

Theorem 2.4. *Suppose that $S \subset \mathbb{C}$ is a simple closed curve and $f : T(S) \to \mathbb{C}$ is a continuous map such that $\mathrm{Ind}(\mathrm{f}, \mathrm{S}) \neq 0$. Then f has a fixed point in $T(S)$.*

Theorem 2.4 applies to Problem 1.1 as follows. Given a non-separating continuum $X \subset \mathbb{C}$ one constructs a simple closed curve S approximating X so that the index of f on S can be computed. If it is not equal to zero, it implies the existence of a fixed point in $T(S)$, and if S is tight enough, in X. Hence our efforts should be aimed at constructing S and computing $\mathrm{Ind}(\mathrm{f}, \mathrm{S})$. One way of doing so is to use Bell's notion of variation which we will now introduce.

Suppose that X is a non-separating plane continuum and S is a simple closed curve such that $X \subset T(S)$ and $S \cap X$ consists of more than one point. Then we will call S a *bumping simple closed curve of X*. Any subarc of S, both of whose endpoints are in X, is called a *bumping arc of X* or a *link of S*. Note that any bumping arc A of X can be extended to a bumping simple closed curve S of X. Hence, every bumping arc has a natural order $<$ inherited from the positive circular order of a bumping simple closed curve S containing A. If $a < b$ are the endpoints of A, then we will often write $A = [a, b]$. Also, by the *shadow $Sh(A)$ of A*, we mean the union of all bounded components of $\mathbb{C} \setminus (X \cup A)$.

The *standard junction J_0* is the union of the three rays $R_i = \{z \in \mathbb{C} \mid z = re^{i\pi/2}, r \in [0, \infty)\}$, $R_+ = \{z \in \mathbb{C} \mid z = re^0, r \in [0, \infty)\}$, $R_- = \{z \in \mathbb{C} \mid z = re^{i\pi}, r \in [0, \infty)\}$, having the origin 0 in common. By U we denote the lower half-plane $\{z \in \mathbb{C} \mid z = x + iy, y < 0\}$. A *junction J_v* is the image of J_0 under any orientation-preserving homeomorphism $h : \mathbb{C} \to \mathbb{C}$ where $v = h(0)$. We will often suppress h and refer to $h(R_i)$ as R_i, and similarly for the remaining rays and the region $h(U)$.

Definition 2.5 (Variation on an arc). Let $f : \mathbb{C} \to \mathbb{C}$ be a map, X be a non-separating plane continuum, $A = [a, b]$ be a bumping subarc of X with $a < b$, $\{f(a), f(b)\} \subset X$ and $f(A) \cap A = \emptyset$. We define the *variation of f on A with respect to X*, denoted $\mathrm{Var}(\mathrm{f}, \mathrm{A})$, by the following algorithm:

(1) Choose an orientation preserving homeomorphism h of $\mathbb{C}$ such that $h(0) = v \in A$ and $X \subset h(U) \cup \{v\}$.

crossings

(2) *Crossings:* Consider the set $K = [a, b] \cap f^{-1}(J_v)$. Move along A from a to b. Each time a point of $[a, b] \cap f^{-1}(R_+)$ is followed immediately by a point of $[a, b] \cap f^{-1}(R_i)$ in K, count $+1$. Each time a point of $[a, b] \cap f^{-1}(R_i)$ is followed immediately by a point of $[a, b] \cap f^{-1}(R_+)$ in K, count -1. Count no other crossings.

(3) The sum of the crossings found above is the variation, denoted $\mathrm{Var}(\mathrm{f}, \mathrm{A})$.

It is shown in [FMOT07] that the variation does not depend on the choice of a junction satisfying the above listed properties. Informally, one can understand the notion of variation as follows. Since $f(A) \cap A = \emptyset$, we can always complete A with

another arc B (now connecting b to a) to a simple closed curve S disjoint from J_v so that $v \notin f(S)$. Then it is easy to see that $\text{win}(f, S, v)$ cab be obtained by summing up $\text{Var}(f, A)$ and the similar count for the arc B (observe that the latter is *not* the variation of B because to compute that we will need to use another junction "based" at a point of B).

Any partition $A = \{a_0 < a_1 < \cdots < a_n < a_{n+1} = a_0\} \subset X \cap S$ of a bumping simple closed curve S of a non-separating continuum X such that for all i, $f(a_i) \in X$ and $f([a_i, a_{i+1}]) \cap [a_i, a_{i+1}] = \emptyset$ is called an *allowable partition of S*. We will call the bumping arcs $[a_i, a_{i+1}]$ *links (of S)*. It is shown in [FMOT07] that the variation of a bumping arc is well-defined. Moreover, it follows from that paper (see Theorem 2.12 and Remark 2.19) that:

Theorem 2.6. *Let S be a simple closed curve, $X = T(S)$ and let $a_0 < a_1 < \cdots < a_n < a_0 = a_{n+1}$ be points in S (in the positive circular order around S) such that for each i, $f(a_i) \in T(S)$ and, if $Q_i = [a_i, a_{i+1}]$, then $f(Q_i) \cap Q_i = \emptyset$.*
Then

$$\text{Ind}(f, S) = \sum_i \text{Var}(f, Q_i) + 1.$$

Observe that if the points $a_i, i = 1, \ldots, n$ satisfying the properties of Theorem 2.6 can be chosen then there are no fixed points of f in S and $\text{Ind}(f, S)$ is well-defined. Theorem 2.6 shows that if we define $\text{Var}(f, S) = \sum_i \text{Var}(f, Q_i)$, then $\text{Var}(f, S)$ is well defined and independent of the choice of the allowable partition of S and of the choice of the junctions used to compute $\text{Var}(f, Q_i)$.

Lemma 2.7. *Let $f : \mathbb{C} \to \mathbb{C}$ be a map, X be a non-separating continuum and $C = [a, b]$ be a bumping arc of X with $a < b$. Let $v \in [a, b]$ be a point and let J_v be a junction such that $J_v \cap (X \cup C) = \{v\}$. Suppose that $J_v \cap f(X) = \emptyset$. Then there exists an arc I such that $S = I \cup C$ is a bumping simple closed curve of X and $f(I) \cap J_v = \emptyset$.*

Proof. Since $f(X) \cap J_v = \emptyset$, it is clear that there exists an arc I with endpoints a and b near X such that $I \cup C$ is a simple closed curve, $X \subset T(I \cup C)$ and $f(I) \cap J_v = \emptyset$. This completes the proof. $\square$

The next corollary gives a sufficient condition for the non-negativity of the variation of an arc.

Corollary 2.8. *Suppose $f : \mathbb{C} \to \mathbb{C}$ is a positively oriented map, $C = [a, b]$ is a bumping arc of X, $f(C) \cap C = \emptyset$ and $J_v \subset [\mathbb{C} \setminus X] \cup \{v\}$ is a junction with $J_v \cap C = \{v\}$. Suppose that $f(\{a, b\}) \subset X$ and there exists a continuum $K \subset X$ such that $f(K) \cap J_v \subset \{v\}$. Then $\text{Var}(f, C) \geq 0$.*

Proof. Observe that since $f(a), f(b) \in X$ then $\text{Var}(f, C)$ is well-defined. Consider a few cases. Suppose first that $f(K) \cap J_v = \emptyset$. Then, by Lemma 2.7, there exists an arc I such that $S = I \cup C$ is a bumping simple closed curve around K and $f(I) \cap J_v = \emptyset$ (it suffices to choose I very close to K). Then $v \in \mathbb{C} \setminus f(S)$. Hence $\text{Var}(f, C) = \text{win}(f, S, v) \geq 0$. Suppose next that $f(K) \cap J_v = \{v\}$. Then we can perturb the junction J_v slightly in a small neighborhood of v, obtaining a new junction J_d such that intersections of $f(C)$ with J_v and J_d are the same (and, hence, yield the same variation) and $f(K) \cap J_d = \emptyset$. Now proceed as in the first case. $\square$

3. MAIN RESULTS

3.1. **Dendrites.** In this subsection we generalize Theorem 1.2 to non-invariant dendrites. We will also show that in certain cases the dendrite must contain infinitely many periodic cutpoints (recall that if Y is a continuum and $x \in Y$ then $\mathrm{val}_Y(x)$ is the number of connected components of $Y \setminus \{x\}$, and x is said to be an *endpoint (of Y)* if $\mathrm{val}_Y(x) = 1$, a *cutpoint (of Y)* if $\mathrm{val}_Y(x) > 1$ and a *vertex/branchpoint (of Y)* if $\mathrm{val}_Y(x) > 2$). These results have applications in complex dynamics [BCO08]. In this subsection given two points a, b of a dendrite we denote by $[a, b], (a, b], [a, b), (a, b)$ the unique closed, semi-open and open arcs connecting a and b in the dendrite. More specifically, unless otherwise specified, the situation considered in this subsection is as follows: $D_1 \subset D_2$ are dendrites and $f : D_1 \to D_2$ is a continuous map. Set $E = \overline{D_2 \setminus D_1} \cap D_1$. In other words, E consists of points at which D_2 "grows" out of D_1. A point $e \in E$ may be an endpoint of D_1 (then there is a unique component of $D_2 \setminus \{e\}$ which meets D_1) or a cutpoint of D_1 (then there are several components of $D_2 \setminus \{e\}$ which meet D_1).

The following theorem is a simple extension of the real result claiming that if there are points $a < b$ in $\mathbb{R}$ such that $f(a) < a, f(b) > b$ then there exists a fixed point $c \in (a, b)$ (case (b) described in Introduction).

Proposition 3.1. *Suppose that $a, b \in D_1$ are such that a separates $f(a)$ from b and b separates $f(b)$ from a. Then there exists a fixed point $c \in (a, b)$ which is a cutpoint of D_1 (and hence D_2). In particular if there are two points $e_1 \neq e_2 \in E$ such that $f(e_i)$ belongs to a component of $D_2 \setminus \{e_i\}$ disjoint from D_1 then there exists a fixed point $c \in (a, b)$ which is a cutpoint of D_1 (and hence D_2).*

Proof. It follows that we can find a sequence of points $a_{-1}, \ldots$ in (a, b) such that $f(a_{-n-1}) = a_{-n}$ and a_{-n-1} separates a_{-n} from b. Clearly, $\lim_{n \to \infty} a_{-n} = c \in [a, b]$ is a fixed point as desired. If there are two points $e_1 \neq e_2 \in E$ such that $f(e_i)$ belongs to a component of $D_2 \setminus \{e_i\}$ disjoint from D_1 then the above applies to them. $\square$

The other real case (case (a)) described in Introduction is somewhat more difficult to generalize. Definition 3.2 extends it (i.e. the real case when $a < b$ are points of $\mathbb{R}$ such that $f(a) > a$ and $f(b) < b$) onto dendrites.

Definition 3.2. Suppose that in the above situation the map f is such that for each non-fixed point $e \in E$, $f(e)$ is contained in a component of $D_2 \setminus \{e\}$ which meets D_1. Then we say that f has the *boundary scrambling property* or that it *scrambles the boundary*. Observe that if D_1 *is* invariant then f automatically scrambles the boundary.

The next definition presents a useful topological version of repelling at a fixed point.

Definition 3.3. Suppose that $a \in D_1$ is a fixed point and that there exists a component B of $D_1 \setminus \{a\}$ such that arbitrarily close to a in B there exist fixed cutpoints of f or points x separating p from $f(x)$. Then say that a is a *weakly repelling fixed point (of f in B)*. A periodic point a is said to be *weakly repelling* if there exists n and a component B of $D_1 \setminus \{a\}$ such that a is a weakly repelling fixed point of f^n in B.

It is easy to see that a fixed point a is weakly repelling in B if and only if either a is a limit of fixed cutpoints of f in B, or there exist a neighborhood U of a in $\overline{B}$ and a point $x \in U \setminus \{a\}$ such that U contains no fixed points but a and x separates p from $f(x)$. Indeed, in the latter case by continuity there exists a point $x_1 \in (a, x)$ such that $f(x_{-1}) = x$ and this sequence of preimages can be extended towards a inside (a, x) so that it converges to a (otherwise it would converge to a fixed point inside U distinct from a, a contradiction). In particular, *if a is a weakly repelling fixed point of f then a is a weakly repelling fixed point of f^n for any n.* Moreover, since there are only countably many vertices of D_2 and their images under f and its powers, we can choose x and its backward orbit converging to a so that all its points are cutpoints of D_2 of valence 2. From now on we assume that to each weakly repelling fixed point a of f in B which is not a limit point of fixed cutpoints in B we associate a point $x_a = x \in B$ of valence 2 in B separating a from $f(x)$ and a neighborhood $U_a = U \subset \overline{B}$ which is the component of $\overline{B} \setminus \{x\}$ containing a.

As an important tool we will need the following retraction closely related to the described above situation.

retr **Definition 3.4.** For each $x \in D_2$ there exists a unique arc (possibly a point) $[x, d_x]$ such that $[x, d_x] \cap D_1 = \{d_x\}$. Hence there exists a natural monotone retraction $r : D_2 \to D_1$ defined by $r(x) = d_x$, and the map $g = g_f = r \circ f : D_1 \to D_1$ which is a continuous map of D_1 into itself. We call the map r the *natural retraction (of D_2 onto D_1)* and the map g the *retracted (version of) f.*

The map g is designed to make D_1 invariant so that Theorem 1.2 applies to g and allows us to conclude that there are g-fixed points. However these points are not necessarily fixed points of f. Indeed, $g(x) = x$ means that $r \circ f(x) = x$. Hence it really means that f maps x to a point belonging to a component of $D_2 \setminus D_1$ which "grows" out of D_1 at x. In particular, it means that $x \in E$. Thus, if $g(x) = x$ and $x \notin E$ then $f(x) = x$. In general, it follows from the construction that if $f(x) \neq g(x)$, then $g(x) \in E$ because points of E are exactly those points of D_1 to which points of $D_2 \setminus D_1$ map under r. We are ready to prove our first lemma in this direction.

fixpt0 **Proposition 3.5.** *Suppose that f scrambles the boundary. Then f has a fixed point.*

Proof. We may assume that there are no f-fixed points $e \in E$. By Theorem 1.2 the map $g_f = g$ has a fixed point $p \in D_1$. It follows from the fact that f scrambles the boundary that points of E are not g-fixed. Hence $p \notin E$, and by the argument right before the lemma $f(p) = p$ as desired. $\square$

It follows from Proposition 3.1 and Proposition 3.5 that the only behavior of points in E which does not force the existence of a fixed point in D_1 is when exactly one point $e \in E$ maps into a component of $D_2 \setminus \{e\}$ which is disjoint from D_1 whereas any other point $e' \in E$ maps into a component of $D_2 \setminus \{e'\}$ which is not disjoint from D_1.

The next lemma shows that in some cases p can be chosen to be a cutpoint of D_1.

fxctpt **Lemma 3.6.** *Suppose that f scrambles the boundary and all f-fixed endpoints of D_1 are weakly repelling. Then there is a fixed cutpoint of f.*

Proof. Suppose that f has no fixed cutpoints. By Proposition 3.5, the set of fixed points of f is not empty. Hence we may assume that *all* fixed points of f are endpoints

of D_1. Let a, b be distinct fixed points of f. Then it follows that either $U_a \subset U_b$, or $U_b \subset U_a$, or $U_a \cap U_b = \emptyset$. Indeed, suppose that $x_a \in U_b$. Let us show that then $U_a \subset U_b$. Indeed, otherwise by Lemma 3.5.(1) there exists a fixed point $c \in (x_a, x_b)$, a contradiction. Now, suppose that neither $x_a \in U_b$ nor $x_b \in U_a$. Then clearly $U_a \cap U_b = \emptyset$. Consider an open covering of the set of all fixed points $a \in D_1$ by their neighborhoods U_a and choose a finite subcover. By the above we may assume that its consists of pairwise disjoint sets $U_{a_1}, \ldots, U_{a_k}$. Consider now the component Q of D_1 whose endpoints are the points $a_1, \ldots, a_k$. It is easy to see that $f|_Q$, with Q considered as a subdendrite of D_2, scrambles the boundary and has no fixed endpoints. Hence an f-fixed point $p \in Q$, existing by Lemma 3.5, must be a cutpoint of D_1. $\square$

Lemma 3.6 is helpful in the next theorem which shows that under some rather weak assumptions on periodic points the map has infinitely many periodic cutpoints.

Theorem 3.7. *Suppose that $f : D \to D$ is continuous and all its periodic points are weakly repelling. Then f has infinitely many periodic cutpoints.*

Proof. By way of contradiction suppose that there are finitely many periodic cutpoints of f. Without loss of generality we may assume that these are points $a^1, \ldots, a^k$ each of which is *fixed* under f. Let $A = \cup_{i=1}^k a^i$ and let B be component of $D \setminus A$. Then $\overline{B}$ is a subdendrite of D to which the above tools apply: D plays the role of D_2, $\overline{B}$ plays the role of D_1, and E is exactly the boundary $\mathrm{Bd}(B)$ of B (by the construction $\mathrm{Bd}(B) \subset A$). Suppose that each point $a \in \mathrm{Bd}(B)$ is weakly repelling in B. Then by the assumptions of the theorem Lemma 3.6 applies to this situation. It follows that there exists a fixed cutpoint of B, a contradiction. Hence for some $a \in \mathrm{Bd}B$ we have that a is *not* weakly repelling in B. Since by the assumptions a *is* weakly repelling, there exists another component, say, C, of $D \setminus A$ such that $a \in \mathrm{Bd}(C)$ and a *is* weakly repelling in C.

We can now apply the same argument to C. If all boundary points of C are weakly repelling in C then by Lemma 3.6 C will contain a fixed cutpoint, a contradiction. Hence there exists a point $d \in A$ such that d is *not* weakly repelling in C and a component F of $D \setminus A$ whose closure meets $\overline{C}$ at d, and d *is* weakly repelling in F. Clearly, after finitely many steps this process will have to end ultimately leading to a component Z of $D \setminus A$ such that all points of $\mathrm{Bd}(Z)$ are weakly repelling in Z. Since here the set $\mathrm{Bd}(Z)$ plays the role of the set E from above and by the assumptions of the theorem we see that Lemma 3.6 applies to Z and there exists a fixed cutpoint of Z, a contradiction. $\square$

An important application of Theorem 3.7 is to the dendritic *topological Julia sets*. They can be defined as follows. Consider an equivalence relation $\sim$ on the unit circle $\mathbb{S}^1$. Equivalence classes of $\sim$ will be called *($\sim$-)classes* and will be denoted by boldface letters. A $\sim$-class consisting of two points is called a *leaf*; a class consisting of at least three points is called a *gap* (this is more restrictive than Thurston's definition in [Thu85]; we follow [BL02] in our presentation). Fix an integer $d > 1$ and denote the map $z^d : \mathbb{S}^1 \to \mathbb{S}^1$ by σ_d. Then $\sim$ is said to be a *(d-)invariant lamination* if:
(E1) $\sim$ is *closed*: the graph of $\sim$ is a closed set in $\mathbb{S}^1 \times \mathbb{S}^1$;
(E2) $\sim$ defines a *lamination*, i.e., it is *unlinked*: if $\mathbf{g}_1$ and $\mathbf{g}_2$ are distinct $\sim$-classes, then their convex hulls $\mathrm{Ch}(\mathbf{g}_1), \mathrm{Ch}(\mathbf{g}_2)$ in the unit disk $\mathbb{D}$ are disjoint,
(D1) $\sim$ is *forward invariant*: for a class $\mathbf{g}$, the set $\sigma_d(\mathbf{g})$ is a class too

which implies that

(D2) $\sim$ is *backward invariant*: for a class $\mathbf{g}$, its preimage $\sigma_d^{-1}(\mathbf{g}) = \{x \in \mathbb{S}^1 : \sigma_d(x) \in \mathbf{g}\}$ is a union of classes;

(D3) for any gap $\mathbf{g}$, the map $\sigma_d|_{\mathbf{g}} : \mathbf{g} \to \sigma_d(\mathbf{g})$ is a *covering map with positive orientation*, i.e., for every connected component (s, t) of $\mathbb{S}^1 \setminus \mathbf{g}$ the arc $(\sigma_d(s), \sigma_d(t))$ is a connected component of $\mathbb{S}^1 \setminus \sigma_d(\mathbf{g})$.

The lamination in which all points of $\mathbb{S}^1$ are equivalent is said to *degenerate*. It is easy to see that if a forward invariant lamination $\sim$ has a class with non-empty interior then $\sim$ is degenerate. Hence equivalence classes of any non-degenerate forward invariant lamination are totally disconnected.

Call a class $\mathbf{g}$ *critical* if $\sigma_d|_{\mathbf{g}} : \mathbf{g} \to \sigma(\mathbf{g})$ is not one-to-one, *(pre)critical* if $\sigma_d^j(\mathbf{g})$ is critical for some $j \geq 0$, and *(pre)periodic* if $\sigma_d^i(\mathbf{g}) = \sigma_d^j(\mathbf{g})$ for some $0 \leq i < j$. A gap $\mathbf{g}$ is *wandering* if $\mathbf{g}$ is neither (pre)periodic nor (pre)critical. Let $p : \mathbb{S}^1 \to J_\sim = \mathbb{S}^1 / \sim$ be the quotient map of $\mathbb{S}^1$ onto its quotient space $J_\sim$, let $f_\sim : J_\sim \to J_\sim$ be the map induced by σ_d. We call $J_\sim$ a *topological Julia set* and the induced map $f_\sim$ a *topological polynomial*. It is easy to see that if $\mathbf{g}$ is a $\sim$-class then $\mathrm{val}_{J_\sim}(p(\mathbf{g})) = |\mathbf{g}|$ where by $|A|$ we denote the cardinality of a set A.

`lamwkrp` **Theorem 3.8.** *Suppose that the topological Julia set $J_\sim$ is a dendrite and $f_\sim : J_\sim \to J_\sim$ is a topological polynomial. Then all periodic points of $f_\sim$ are weakly repelling and $f_\sim$ has infinitely many periodic cutpoints.*

Proof. Suppose that x is an $f_\sim$-fixed point and set $\mathbf{g} = p^{-1}(x)$. If x is an endpoint then $\mathbf{g}$ is a singleton. Connect x with a point $y \neq x$. Then the arc $[x, y] \subset J_\sim$ contains points $y_k \to x$ of valence 2 because, as is well known, there are no more than countably many vertices of $J_\sim$. It follows that $\sim$-classes $p^1(y_k)$ are leaves separating $\mathbf{g}$ from the rest of the circle and repelled from $\mathbf{g}$ under the action of σ. Hence $f_\sim(y_i)$ is separated from x by y_i and so x is weakly repelling.

Suppose that x is not an endpoint. Choose a very small connected neighborhood U of x. It is easy to see that each component A of $U \setminus \{x\}$ corresponds to a unique chord $\ell_A \in \mathrm{Bd}(\mathrm{Ch}(\mathbf{g}))$. Moreover, for each component A of $U \setminus \{x\}$ there exists a unique component B of $U \setminus \{x\}$ such that $f_\sim(A) \cap B \neq \emptyset$. Hence there is a map h from the set $\mathcal{A}$ of all components of $U \setminus \{x\}$ to itself. Suppose that there exists $E \in \mathcal{A}$ and $n > 0$ such that $h^n(E) = E$. Then it follows that the endpoints of ℓ_E are fixed under σ^n. Connect x with a point $y \in E$ and choose, as in the previous paragraph, a sequence of points $y_k \in [x, y], y_k \to x$ of valence 2. Then again by the repelling properties of σ^n it follows that $f_\sim(y_i)$ is separated from x by y_i and so x is weakly repelling (for $f_\sim^m$ in E).

It remains to show that there $E \in \mathcal{A}$ with $h^n(E) = E$ for some $n > 0$ must exist. Suppose otherwise. To each component C of $U \setminus \{x\}$ we associate the corresponding component J_C of $J_\sim \setminus \{a\}$ containing C. Then there are only finitely many such components C of $U \setminus \{x\}$ that J_C contains a critical point; denote their collection by $\mathcal{C}$. Let us show that an eventual h-image of every $E \in \mathcal{A}$ must coincide with an element of $\mathcal{C}$. Indeed, otherwise there is a component $E \in \mathcal{A}$ such that all $h^k(E)$ are distinct and the map $f_\sim|_{J_{h^k(E)}}$ is a homeomorphism. Clearly, this implies the existence of a wandering subcontinuum K of $J_\sim$. However by Theorem C [BL02] this is impossible.

Hence all periodic points of $f_\sim$ are weakly repelling and by Theorem 3.7 $f_\sim$ has infinitely many periodic cutpoints. $\square$

3.2. **Positively oriented maps of the plane.** In this subsection we will first obtain a general fixed point theorem which shows that if a non-separating plane continuum, not necessarily invariant, maps in an appropriate way, then it must contain a fixed point. This extends Theorem 3.9. Let us denote the family of all positively oriented maps of the plane by $\mathcal{P}$.

fmot1

Theorem 3.9 ([FMOT07])**.** *Suppose that $f \in \mathcal{P}$ and that $X \subset \mathbb{C}$ is a non-separating continuum such that $f(X) \subset X$. Then there exists a fixed point $p \in X$.*

To proceed we will need to generalize Corollary 2.8 to a more general situation. To this end we introduce a definition similar to the one given for dendrites in the previous section.

scracon

Definition 3.10. Suppose that $f \in \mathcal{P}$ and X is non-separating continuum. Suppose that there exist $n \geq 0$ disjoint non-separating continua Z_i such that the following properties hold:

(1) $f(X) \setminus X \subset \cup_i Z_i$;
(2) for all i, $Z_i \cap X = K_i$ is a non-separating continuum;
(3) for all i, $f(K_i) \cap [Z_i \setminus K_i] = \emptyset$.

Then the map f is said to *scramble the boundary (of X)*. If instead of (3) we have

(3a) for all i, either $f(K_i) \subset K_i$, or $f(K_i) \cap Z_i = \emptyset$

then we say that f *strongly scrambles the boundary (of X)*. In either case, K_i's are called *exit continua (of X)*. Note that since $Z_i \cap X$ is a continuum, $X \cup (\bigcup Z_i)$ is a non-separating continuum. Speaking of maps which (strongly) scramble the boundary, we always use the notation from this definition unless explicitly stated otherwise.

Observe that in the situation of Definition 3.10 if X is invariant then f automatically strongly scrambles the boundary because we can simply take the set of exit continua to be empty. Also, if f strongly scrambles the boundary of X and $f(K_i) \not\subset K_i$ for any i, then it is easy to see that there exists $\varepsilon > 0$ such that for every point $x \in X$ either $d(x, Z_i) > \varepsilon$, or $d(f(x), Z_i) > \varepsilon$. Let us now prove the following technical lemma.

posvar+

Lemma 3.11. *Suppose that $f : \mathbb{C} \to \mathbb{C}$ scrambles the boundary of X. Let Q be a bumping arc of X with endpoints $a < b \in X$ such that $f(\{a, b\}) \subset X$ and $f(Q) \cap Q = \emptyset$. Then $\mathrm{Var(f, Q)} \geq 0$.*

Proof. We will use the notation as specified in the lemma. Suppose first that $Q \setminus \bigcup Z_i \neq \emptyset$ and choose $v \in Q \setminus \bigcup Z_i$. Since $v \in Q \setminus \bigcup Z_i$ and $X \cup (\bigcup Z_i)$ is non-separating, there exists a junction J_v, with $v \in Q$, such that $J_v \cap [X \cup Q \cup \bigcup Z_i] = \{v\}$ and, hence, $J_v \cap f(X) \subset \{v\}$. Now the desired result follows from Corollary 2.8.

Observe that if $Q \setminus \cup Z_i = \emptyset$ then $Q \subset Z_i$ for some i and so $Q \cap X \subset K_i$. In particular, both endpoints a, b of Q are contained in K_i. Choose a point $v \in Q$. Then again there is a junction connecting v and infinity outside X (except possibly for v). Since all sets $Z_j, j \neq i$ are positively distant from v and $X \cup (\bigcup_{i \neq j} Z_i)$ is non-separating, the junction J_v can be chosen to avoid all sets $Z_j, j \neq i$. Now, by (3) $f(K_i) \cap J_v \subset \{v\}$, hence by Lemma 2.8 $\mathrm{Var(f, Q)} \geq 0$. $\square$

Lemma 3.11 is applied in Theorem 3.12 in which we show that a map which strongly scrambles the boundary has fixed points. In fact, it is a major technical tool in our other results too. Indeed, if we can construct a bumping simple closed curve S around X such that the endpoints of its links map back into X while these links move completely off themselves, the lemma would imply that the variation of S is non-negative. By Theorem 2.6 this would imply that the index of S is positive. Hence by Theorem 2.4 there are fixed points in $T(S)$. Choosing S to be sufficiently tight we see that there are fixed points in X.

For the sake of convenience we now sketch the proof of Theorem 3.12 which allows us to emphasize the main ideas rather than details. The main steps in constructing S are as follows. First we assume by way of contradiction that a map $f : \mathbb{C} \to \mathbb{C}$ has no fixed points in X. Then by Theorem 3.9 it implies that $f(X) \not\subset X$ and that $f(K_i) \not\subset K_i$ for any i. By the definition of strong scrambling then $f(K_i)$ is far away from Z_i for any i. Now, since there are no fixed points in X we can choose the links in S to be very small so that they will all move off themselves. However some of them will have endpoints mapping outside X which prevents us from directly applying Lemma 3.11 to them. These links will be enlarged by concatenating them so that the images of the endpoints of these concatenations are inside X *and* these concatenations still map off themselves. The bumping simple closed curve S then remains as before, however the representation of S as the union of links changes because we enlarge some of them. Still, the construction shows that Lemma 3.11 applies to the new "bigger" links and as before this implies the existence of a fixed point in X.

To achieve the goal of replacing some links in S by their concatenations we consider the links which are mapped outside X in detail using the fact that f strongly scrambles the boundary (indeed, all other links are such that Lemma 3.11 already applies to them). The idea is to consider the links of S whose concatenation is a connected piece of S mapping into one Z_i. Then if we begin the concatenation right before the images of links enter Z_i and stop it right after the images of the links exit Z_i we will have one condition of Lemma 3.11 satisfied because the endpoints of the thus constructed new "big" concatenation link T of S map into X.

We now need to verify that T moves off itself under f. Indeed, this is easy to see for the end-links of T: each end-link has the image "crossing" into Z_i from $X \setminus Z_i$, hence the images of end-links are close to K_i. However the sets K_i are mapped far away from Z_i by the definition of strong scrambling and because none of K_j's is invariant by the assumption. This implies that the end-links themselves must be far away from Z_i. If now we move from link to link inside T we see that those links cannot approach Z_i too closely because if they do they will have to "cross over K_i" into Z_i, and then their images will have to be close to the image of K_i which is far away from Z_i, a contradiction with the fact that all links in T have endpoints which map into Z_i. In other words, the dynamics of K_i prevents the new bigger links from getting even close to Z_i under f which shows that they move off themselves as desired. As before, we now apply Theorem 2.6 to see that $\mathrm{Ind}(f, S) = \mathrm{Var}(f, S) + 1$ and then Theorem 2.4 to see that this implies the existence of a fixed point in X.

Given a compact set K denote by $B(K, \varepsilon)$ the open set of all points whose distance to K is less than ε.

Theorem 3.12. *Suppose that $f : \mathbb{C} \to \mathbb{C}$ strongly scrambles the boundary of X. Then f has a fixed point in X.*

Proof. If $f(X) \subset X$ then the result follows from [FMOT07]. Similarly, if there exists i such that $f(K_i) \subset K_i$, then f has a fixed point in $T(K_i) \subset X$ and we are also done. Hence we may assume $f(X) \setminus X \neq \emptyset$ and $f(K_i) \cap Z_i = \emptyset$ for all i. Suppose that f is fixed point free. Then there exists $\varepsilon > 0$ such that for all $x \in X$, $d(x, f(x)) > \varepsilon$. We may assume that $\varepsilon < \min\{d(Z_i, Z_j) \mid i \neq j\}$. We now choose constants η', η, δ and a bumping simple closed curve S of X so that the following holds.

(1) $0 < \eta' < \eta < \delta < \varepsilon/3$.
(2) For each $x \in X \cap B(K_i, 3\delta)$, $d(f(x), Z_i) > 3\delta$.
(3) For each $x \in X \setminus B(K_i, 3\delta)$, $d(x, Z_i) > 3\eta$.
(4) For each i there exists a point $x_i \in X$ such that $f(x_i) = z_i \in Z_i$ and $d(z_i, X) > 3\eta$.
(5) $X \subset T(S)$ and $A = X \cap S = \{a_0 < a_1 < \cdots < a_n < a_{n+1} = a_0\}$ in the positive circular order around S.
(6) $f|_{T(S)}$ is fixed point free.
(7) For the closure $Q_i = [a_i, a_{i+1}]$ of a component of $S \setminus X$, $\mathrm{diam}(Q) + \mathrm{diam}(f(Q)) < \eta$.
(8) For any two points $x, y \in X$ with $d(x, y) < \eta'$ we have $d(f(x), f(y)) < \eta$.
(9) A is an η'-net in $\mathrm{Bd}(X)$.

Observe that by the triangle inequality, $Q_i \cap f(Q_i) = \emptyset$ for every i.

Claim 1. *There exists a point a_j such that $f(a_j) \in X \setminus \cup \overline{B(Z_i, \eta)}$.*
Proof of Claim 1. Set $\overline{B(Z_i, 3\eta)} = T_i$ and show that there exists a point $x \in \mathrm{Bd}(X)$ with $f(x) \in X \setminus \cup T_i$. Indeed, suppose first that $n = 1$. Then $f(K_1) \subset X \setminus T_1$ and we can choose any point of $K_1 \cap \mathrm{Bd}(X)$ as x. Now, suppose that $n \geq 2$. Observe that the sets T_i are pairwise disjoint compacta. By Theorem 2.3 $f(\mathrm{Bd}(X)) \supset \mathrm{Bd}(f(X))$. Hence there are points $x_1 \neq x_2$ in $\mathrm{Bd}(X)$ such that $f(x_1) \in Z_1 \subset T_1, f(x_2) \in Z_2 \subset T_2$. Since the sets $f^{-1}(T_i) \cap X$ are pairwise disjoint non-empty compacta we see that the set $V = \mathrm{Bd}(X) \setminus \cup f^{-1}(T_i)$ is non-empty (because $\mathrm{Bd}(X)$ is a continuum). Now we can choose any point of V as x.

It remains to notice that by the choice of A we can find a point a_j such that $d(a_j, x) < \eta'$ which implies that $d(f(a_j), f(x)) < \eta$ and hence $f(a_j) \in X \setminus \cup \overline{B(Z_i, \eta)}$ as desired. $\qquad\square$

There exists a point x_1 such that $f(x_1) = z_1$ is more than 3η-distant from X. We can find $a \in A$ such that $d(a, x_1) < \eta'$ and hence $f(a) \notin X$ is at least 2η-distant from X. On the other hand, by Claim 1 there are points of A mapped into X. Let $<$ denote the circular order on the set $\{0, 1, \ldots, n+1\}$ defined by $i < j$ if $a_i < a_j$ in the positive circular order around S. Then we can find $n(1) < m(1)$ such that the following claims hold.

(1) $f(a_{n(1)-1}) \in X \setminus \cup Z_i$.
(2) $f(a_r) \in f(X) \setminus X$ for all r with $n(1) \leq r \leq m(1) - 1$ (and so, since $\mathrm{diam}(f(Q_u)) < \varepsilon/3$ for any u and $d(Z_s, Z_t) > \varepsilon$ for all $s \neq t$, there exists $i(1)$ with $f(a_r) \in Z_{i(1)}$ for all $n(1) \leq r < m(1)$).
(3) $f(a_{m(1)} \in X \setminus \cup Z_i$.

Consider the arc $Q'_1 = [a_{n(1)-1}, a_{m(1)}] \subset S$ and show that $f(Q'_1) \cap Q'_1 = \emptyset$. As we walk along Q'_1, we begin outside $Z_{i(1)}$ at $f(a_{n(1)-1})$, then enter $Z_{i(1)}$ and walk inside it, and then exit $Z_{i(1)}$ at $f(a_{m(1)})$. Since every step in this walk is rather short $(\mathrm{diam}(Q_i) + \mathrm{diam}(f(Q_i)) < \eta)$, we see that $d(f(a_{n(1)-1}), Z_{i(1)}) < \eta$ and $d(f(a_{m(1)}), Z_{i(1)}) < \eta$. On the other hand for each $r, n(1) \leq r < m(1)$ we have $f(a_r) \in Z_{i(1)}$, hence we see that $d(f(a_r), Z_{i(1)}) < \eta$ for each $r, n(1) \leq r < m(1)$. This implies that $d(a_r, K_{i(1)}) > 3\delta$ (because otherwise $f(a_r)$ would be farther away from $Z_{i(1)}$) and so $d(a_r), Z_{i(1)}) > 3\eta$ (because $a_r \in X \setminus B(K_{i(1)}, 3\delta)$). Since $\mathrm{diam}(Q) + \mathrm{diam}(f(Q)) < \eta$, then $d(Q'_1, Z_{i(1)}) > 2\delta > 2\eta$. On the other hand, $d(f(a_r), Z_{i(1)}) < \eta$ similarly implies that $d(f(Q'_1), Z_{i(1)}) < 2\eta$. Thus indeed $f(Q'_1) \cap Q'_1 = \emptyset$.

This allows us to replace the original division of S into its prime links $Q_1, \ldots, Q_n$ by a new one in which Q'_1 plays the role of a new prime link; in other words, we simply delete the points $\{a_{n(1)}, \ldots, a_{m(1)-1}\}$ from A. Continuing in the same manner and moving along S, in the end we obtain a finite set $A' = \{a_0 = a'_0 < a'_1 < \cdots < a'_k\}$ such that for each i we have $f(a'_i) \in X \subset T(S)$ and for each arc $Q'_i = [a'_i, a'_{i+1}]$ we have $f(Q'_i) \cap Q'_i = \emptyset$. Hence, by Theorem 2.6, $\mathrm{Ind}(f, S) = \sum_{Q'_i} \mathrm{Var}(f, Q'_i) + 1$. Since by Lemma 3.11, $\mathrm{Var}(f, Q'_i) \geq 0$ for all i, $\mathrm{Ind}(f, S) \geq 1$ contradicting the fact that f is fixed point free in $T(S)$. $\qquad \square$

3.3. Maps with isolated fixed points.
Now we consider maps $f \in \mathcal{P}$ with isolated fixed points; denote the set of such maps by $\mathcal{P}_i$. We need a few definitions.

<div style="border:1px solid;display:inline-block;padding:2px">crit</div> **Definition 3.13.** Given a map $f : X \to Y$ we say that $c \in X$ is a *critical point of* f if for any neighborhood U of c, there exist $x_1 \neq x_2 \in U$ such that $f(x_1) = f(x_2)$. Hence, if x is not a critical point of f, then f is locally one-to-one near x.

If a point x belongs to a continuum collapsed under f then x is critical; also any point which is an accumulation point of collapsing continua is critical. However in this case the map around x may be monotone. A more interesting case is when the map around x is not monotone; then x is a *branchpoint* of f and it is critical even if there are no collapsing continua close by. One can define the *local degree* $\deg_f(a)$ as the number of components of $f^{-1}(y)$, for a point y close to $f(a)$, which are non-disjoint from a small neighborhood of a. Then branchpoints are exactly the points at which the local degree is more than 1. Notice that since we do not assume any smoothness, a critical point may well be both fixed (periodic) and be such that small neighborhoods of $c = f(c)$ map over themselves by f.

The next definition is closely related to that of the index of the map on a simple closed curve.

<div style="border:1px solid;display:inline-block;padding:2px">indpt</div> **Definition 3.14.** Suppose that x is a fixed point of a map $f \in \mathcal{P}_i$. Then the *local index of f at x*, denoted by $\mathrm{Ind}(f, x)$, is defined as $\mathrm{Ind}(f, S)$ where S is a small simple closed curve around x.

It is easy to see that if $f \in \mathcal{P}_i$, then the local index is well-defined, i.e. does not depend on the choice of S. By modifying a translation map one can give an example of a homeomorphism of the plane which has exactly one fixed point x with local index 0. Still in some cases the local index at a fixed point must be positive.

<div style="border:1px solid;display:inline-block;padding:2px">toprepat</div> **Definition 3.15.** A fixed point x is said to be *topologically repelling* if there exists a sequence of simple closed curves $S_i \to \{x\}$ such that $x \in \mathrm{int}(T(S_i)) \subset T(S_i) \subset$

$\text{int}(T(f(S_i)))$. A fixed point x is said to be *topologically attracting* if there exists a sequence of simple closed curves $S_i \to \{x\}$ not containing x and such that $x \in \text{int}(T(f(S_i)) \subset T(f(S_i)) \subset \text{int}(T(S_i))$.

Lemma 3.16. *If a is a topologically repelling fixed point then* $\text{Ind}(f, a) = \deg_f(a) \geq 1$ *where d is the local degree. If however a is a topologically attracting fixed point then* $\text{Ind}(f, a) = 1$.

Proof. Consider the case of the repelling fixed point a. Then it follows that, as x runs along a small simple closed curve S with $a \in T(S)$, the vector from x to $f(x)$ produces the same winding number as the vector from a to $f(x)$, and it is easy to see that the latter equals $\deg_f(a)$. The argument with attracting fixed point is similar. □

If however x is neither topologically repelling nor topologically attracting, then $\text{Ind}(f, x)$ could be greater than 1 even in the non-critical case. Indeed, take a neutral fixed point of a rational function. Then it follows that if $f'(x) \neq 1$ then $\text{Ind}(f, x) = 1$ while if $f'(x) = 1$ then $\text{Ind}(f, x)$ is the multiplicity at x (i.e., the local degree of the map $f(z) - z$ at x). This is related the following useful theorem. It is a version a more general, topological *argument principle* stated in the convenient for us form.

Theorem 3.17. *Suppose that $f \in \mathcal{P}_i$. Then for any simple closed curve $S \subset \mathbb{C}$ which contains no fixed points of f its index equals the sum of local indices taken over all fixed points in $T(S)$.*

Theorem 3.17 implies Theorem 2.4 but provides more information. In particular if S were a simple closed curve and if we knew that the local index at any fixed point $a \in T(S)$ is 1, it would imply that $\text{Ind}(f, S)$ equals the number $n(f, S)$ of fixed points of f in $T(S)$. By the above analysis this holds if all f-fixed points in $T(S)$ are either repelling, or attracting, or neutral and such that f has a complex derivative f' in a small neighborhood of x, and $f'(x) \neq 1$.

In the spirit of the previous parts of the paper, we are still concerned with finding f-fixed points inside non-invariant continua of which f (strongly) scrambles the boundary. However we now specify the types of fixed points we are looking for. Thus, the main result of this subsection proves the existence of specific fixed points in non-degenerate continua satisfying the appropriate boundary conditions and shows that in some cases such continua must be degenerate. It is in this latter form that we apply the result later on in Section 4.

Given a non-separating continuum X, a ray $R \subset \mathbb{C} \setminus X$ from ∞ which lands on $x \in X$ (i.e., $\overline{R} \setminus R = \{x\}$) and a crosscut Q of X we say that Q and R *cross essentially* provided there exists $r \in R$ such that the subarc $[x, r] \subset \overline{R}$ is contained in $\text{Sh}(Q)$. The next definition complements the previous one.

Definition 3.18. If $f(p) = p$ and $p \in \text{Bd}(X)$ then we say that f *repels outside X at* p provided there exists a ray $R \subset \mathbb{C} \setminus X$ from ∞ which lands on p and a sequence of simple closed curves S^j bounding closed disks D^j such that $D^1 \supset D^2 \supset \ldots$, $\cap D^j = \{p\}$, $f(D^1 \cap X) \subset X$, $f(\overline{S^j \setminus X}) \cap D^j = \emptyset$ and for each j there exists a component Q^j of $S^j \setminus X$ such that $Q^j \cap R \neq \emptyset$ and $\text{Var}(f, Q^j) \neq 0$. If $f \in \mathcal{P}$ and f scrambles the boundary of X, then by Lemma 3.11, for any component Q of $S^j \setminus X$ we have $\text{Var}(f, Q) \geq 0$ so that in this case $\text{Var}(f, Q^j) > 0$.

The next theorem is the main result of this subsection.

$\boxed{\text{locrot}}$ **Theorem 3.19.** *Suppose that $f \in \mathcal{P}_i$, and $X \subset \mathbb{C}$ is a non-separating continuum or a point. Moreover, the following conditions hold.*

$\boxed{\text{fV}}$ (1) *For each fixed point $p \in X$ we have that $\mathrm{Ind}(f, x) = 1$ and f repels outside X at p.*

    (2) *The map f scrambles the boundary of X. Moreover, either $f(K_i) \cap Z_i = \emptyset$, or there exists a neighborhood W_i of K_i with $f(W_i \cap X) \subset X$.*

Then X is a point.

Proof. Suppose that X is not a point. Since $f \in \mathcal{P}_i$, there exists a simply connected neighborhood V of X such that all fixed points $\{p_1, \ldots, p_m\}$ of $f|_{\overline{V}}$ are contained in X. We will show that then f must have at least $m+1$ fixed points in V, a contradiction. The proof will proceed like the proof of Theorem 3.12: we construct a tight bumping simple closed curve S such that $X \subset T(S) \subset V$. We will show that for an appropriate S, $\mathrm{Var}(f, S) \geq m$. Hence $\mathrm{Ind}(f, S) = \mathrm{Var}(f, S) + 1 \geq m+1$ and by Theorem 3.17 f must have at least $m+1$ fixed points in V.

Let us choose neighborhoods U_i of exit continua K_i satisfying conditions listed below.

    (1) For $n_1 < i \leq n$ by assumption (2) of the theorem we may assume that $f(U_i \cap X) \subset X$.

    (2) For $1 \leq i \leq n_1$ we may assume that $d(U_i \cup Z_i, f(U_i)) > 0$.

    (3) We may assume that $T(X \cup \bigcup \overline{U_i}) \subset V$ and $\overline{U_i} \cap \overline{U_k} = \emptyset$ for all $i \neq k$.

    (4) We may assume that every fixed point of f contained in $\overline{U_i}$ is contained in K_i.

Let $\{p_1, \ldots, p_t\}$ be all fixed points of f in $X \setminus \bigcup_i K_i$ and let $\{p_{t+1}, \ldots, p_m\}$ be all the fixed points contained in $\bigcup K_i$. Observe that then by the choice of neighborhoods U_i we have $p_i \in X \setminus \overline{\cup U_s}$ if $1 \leq i \leq t$. Also, it follows that for each $j, t+1 \leq j \leq n$ there exists a unique $r_j, n_1 < r_j \leq n$ such that $p_j \in K_{r_j}$. For each fixed point $p_j \in X$ choose a ray $R_j \subset \mathbb{C} \setminus X$ landing on p_j, as specified in Definition 3.18, and a small simple closed curve S_j bounding a closed disk D_j such that the following claims hold.

    (1) $D_i \cap R_j = \emptyset$ for all $i \neq j$,

    (2) $f(\overline{S_j \setminus X}) \cap D_j = \emptyset$.

    (3) $T(X \cup \bigcup_j D_j) \subset V$.

    (4) $[D_j \cup f(D_j)] \cap [D_k \cup f(D_k)] = \emptyset$ for all $j \neq k$.

    (5) $f(D_j \cap X) \subset X$.

    (6) Denote by $Q(j, s)$ the components of $S_j \setminus X$; then there exists $Q(j, s(j))$, a component of $S_j \setminus X$, with $\mathrm{Var}(f, Q(j, s(j))) > 0$ and $Q(j, s(j)) \cap R_j \neq \emptyset$.

    (7) $[D_j \cup f(D_j)] \cap \bigcup \overline{U_i} = \emptyset$ for all $1 \leq j \leq t$.

    (8) If $t < j \leq n$ then $[D_j \cup f(D_j)] \subset U_{r_j}$.

Note that by (1) for all $i \neq j$, $\mathrm{Sh}(Q(j, s(j)) \cap Q(i, s(i)) = \emptyset$. We need to choose a few constants. First choose $\varepsilon > 0$ such that for all $x \in X \setminus \bigcup D_j$, $d(x, f(x)) > 3\varepsilon$. Then by continuity we can choose $\eta > 0$ such that for each set $H \subset V$ of diameter less than η we have $\mathrm{diam}(H) + \mathrm{diam}(f(H)) < \varepsilon$ and for each crosscut C of X disjoint from $\cup D_j$ we have that $f(C)$ is disjoint from C (observe that outside $\cup D_j$ all points of X move by a bounded away from zero distance). Finally we choose $\delta > 0$ so that the following inequalities hold:

(1) $3\delta < \varepsilon$,
(2) $3\delta < d(Z_i, Z_j)$ for all $i \neq j$,
(3) $3\delta < d(Z_i, [X \cup f(X)] \setminus [Z_i \cup U_i])$,
(4) $3\delta < d(K_i, \mathbb{C} \setminus U_i)$,
(5) if $f(K_i) \cap Z_i = \emptyset$, then $3\delta < d(f(U_i), Z_i \cup U_i)$.

Also, given a crosscut C we can associate to its endpoints external angles α, β whose rays land at these endpoints from the appropriate side of X determined by the location of C (so that the open region of the plane enclosed by a tight equipotential between R_α and R_β, the segments of the rays from the equipotential to the endpoints of C, and C itself, is disjoint from X). Thus we can talk about the angular measure of $Q(j, s(j))$; denote by β the minimum of all such angular measures taken over all crosscuts $Q(j, s(j))$.

Now, choose a bumping simple closed curve S' of X which satisfies two conditions: all its links are (a) less than δ in diameter, and (b) are of angular measures less than β. Clearly this is possible. Then we amend S' as follows. Let us follow S' in the positive direction starting at a link outside $\cup D_j$. Then at some moment for the first time we will be walking along a link of S' which enters some D_j. As it happens, the link L' of S' intersects some $Q(j, s)$ with endpoints a, b and enters the shadow $\text{Sh}(Q(j, s))$. Later on we will be walking outside $\text{Sh}(Q(j, s))$ moving along some link L''. In this case we replace the entire segment of S' from L' to L'' by three links: the first one is a deformation of L' which has the same initial endpoint as L' and the terminal point as a, then $Q(j, s)$, and then a deformation of L'' which begins at b and ends at the same terminal point as L''. In this way we make sure that for all crosscuts $Q(j, s)$ either they are links of S' or they are contained in the shadow of a link of S'. Moreover, by the choice of β crosscuts $Q(j, s(j))$ will have to become links of our new bumping simple closed curve S. By the choice of η and by the properties of crosscuts $Q(j, s)$ it follows that any link of S is disjoint from its image, and for each j, $Q(j, s(j)) \subset S$ and $\text{Var}(f, Q(j, s(j))) > 0$.

We want to compute the variation of S. Each link $Q(j, s(j))$ contributes at least 1 towards $\text{Var}(f, S)$, and we want to show that all other links have non-negative variation. To do so we want to apply Lemma 3.11. Hence we need to verify two conditions on a crosscut listed in Lemma 3.11. One of them follows from the previous paragraph: all links of S move off themselves. However the other condition of Lemma 3.11 may not be satisfied by some links of S because some of their endpoints may map off X. To ensure that for our bumping simple closed curve endpoints e of its links map back into X we have to enlarge links of S and replace some of them by their concatenations (this is similar to what was done in Theorem 3.12). Then we will have to check if the new "bigger" links still have images disjoint from themselves.

Suppose that $X \cap S = A = \{a_0 < a_1 < \cdots < a_n\}$ and $a_0 \in A$ is such that $f(a_0) \in X$ (the arguments similar to those in Theorem 3.12 show that we can may this assumption without loss of generality). Let t' be minimal such that $f(a_{t'}) \notin X$ and $t'' > t'$ be minimal such that $f(a_{t''}) \in X$. Then $f(a_{t'}) \in Z_i$ for some i. Denote by $[a_l, a_r]$ a subarc of S with the endpoints a_r and a_l and moving from a_l to a_r is the positive direction. Since every component of $[a_{t'}, a_{t''}] \setminus X$ has diameter less than δ, $f(a_t) \in Z_i \setminus X$ for all $t' \leq t < t''$. Moreover, for $t' \leq t < t''$, $a_t \notin U_i$. To see this note that if $f(K_i) \cap Z_i = \emptyset$, then by the above made choices $f(U_i) \cap Z_i = \emptyset$, and if

$f(K_i) \cap Z_i \neq \emptyset$, then $f(U_i \cap X) \subset X$ by the assumption. Hence it follows from the property (3) of the constant δ that $f([a_{t'-1}, a_{t''}]) \cap [a_{t'-1}, a_{t''}] = \emptyset$ and we can remove the points a_t, for $t' \leq t < t''$ from the partition A of S. By continuing in the same fashion we obtain a subset $A' \subset A$ such that for the closure of each component C of $S \setminus A'$, $f(C) \cap C = \emptyset$ and for both endpoints a and a' of C, $\{f(a), f(a')\} \subset X$. Moreover, for each j, $Q(j, j(s))$ is a component of $S \setminus A'$.

Now we can apply a variation of the standard argument sketched in Section 2 after Theorem 2.4 and applied in the proof of Theorem 3.12; in this variation instead of Theorem 2.4 we use the fact that f satisfies the argument principle. Indeed, by Theorem 2.6 and Lemma 3.11, $\mathrm{Ind}(f, S) \geq \sum_j \mathrm{Var}(f, Q(j, j(s))) + 1 \geq m + 1$ and by the Theorem 3.17 f has at least $m + 1$ fixed points in $T(S) \subset V$, a contradiction. $\square$

Theorem 3.19 implies the following

Corollary 3.20. *Suppose that f and a non-degenerate X satisfy all the conditions stated in Theorem 3.19. Then either f does not repel outside X at one of its fixed points, or the local index at one of its fixed points is not equal to 1.*

The last lemma of this section gives a sufficient and verifiable condition for a fixed point a belonging to a locally invariant continuum X to be such that the map f repels outside X; we apply the lemma in the next section.

Lemma 3.21. *Suppose that $f : \mathbb{C} \to \mathbb{C}$ is positively oriented, $X \subset \mathbb{C}$ is a continuum and p is a fixed point of f such that:*
   (1) *there exists a neighborhood U of p such that $f|_U$ is one-to-one and $f(U \cap X) \subset X$,*
   (2) *there exists a closed disk $D \subset U$ containing p in its interior such that $f(\partial D) \cap D = \emptyset$ and $\partial D \setminus X$ has at least two components,*
   (3) *there exists a ray $R \subset \mathbb{S} \setminus X$ from infinity such that $\overline{R} = R \cup \{p\}$, $f|_R : R \to R$ is a homeomorphism and for each $x \in R$, $f(x)$ separates x from ∞ in R.*

Then there exists a component C of $\partial D \setminus X$ so that $C \cap R \neq \emptyset$, $\mathrm{Var}(f, C) = +1$ and f repels outside X at p.

Proof. We may assume that $X \setminus U$ contains a continuum. Let $\mathbb{D}^\infty = \mathbb{S} \setminus \overline{\mathbb{D}}$ be the open disk at infinity and let $\varphi : \mathbb{D}^\infty \to \mathbb{S} \setminus X$ be a conformal map such that $\varphi(\infty) = \infty$. Then $T = \varphi^{-1}(R)$ is a ray in $\mathbb{D}^\infty$ which compactifies on a point $\widehat{p} \in S^1$. Let Q_j be all components of $\varphi^{-1}(\partial D \setminus X)$. Then each Q_j is a crosscut of $\mathbb{D}^\infty$. Let $O = \{z \in \mathbb{D}^\infty \mid f \circ \varphi(z) \in \mathbb{S} \setminus X$ and define $F : O \to \mathbb{D}^\infty$ by $F(z) = \varphi^{-1} \circ f \circ \varphi(z)$. Note that $T \cup \bigcup Q_j \subset O$. We may assume that $\overline{Q_1}$ separates $\widehat{p}$ from ∞ in $\overline{\mathbb{D}^\infty}$ and that no other Q_j separates Q_1 from ∞ in $\mathbb{D}^\infty$.

Claim. *$F(Q_1)$ separates Q_1 from ∞ in $\mathbb{D}^\infty$.*

Proof of Claim. Let T_∞ be the component of $T \setminus Q_1$ which contains ∞ and let T_p be the component of $\overline{T} \setminus Q_1$ which contains $\widehat{p}$. Let $a = \overline{T_\infty} \cap Q_1$ and $b = \overline{T_p} \cap Q_1$. Choose a point $b' \in T_p$ very close to b so that the subarc $[b, b'] \subset T_p$ is contained in $\varphi^{-1}(D)$. Let $T'_p \subset T_p$ be the closed subarc from $\widehat{p}$ to b'. Choose an open arc A in the bounded component of $\mathbb{D}^\infty \setminus Q_1$, very close to Q_1 from a to the point $b' \in T_p$ so that $f|_{T'_p \cap A \cup T_\infty}$ is one to-one. Put $Z = T'_p \cap A \cup T_\infty$, then $Q_1 \cap Z = \{a\}$ and $F(Q_1) \cap F(Z) = \{F(a)\}$. Since F is a local orientation preserving homeomorphism

near a, $F(Z)$ enters the bounded component of $\mathbb{D}^\infty \setminus F(Q_1)$ at $F(a)$ and never exits this component after entering it. Moreover, if q is an endpoint of Q_1, then points very close to q on Q_1 and their images are on the same side of T. Since $F(a)$ separates a from ∞ on Z and an initial segment of T_p (with endpoint $\widehat{p}$) is contained in $F(Z)$, $\widehat{p} \in \mathrm{Sh}(F(Q_1))$. This completes the proof of the claim.

Let us compute the variation $\mathrm{Var}(\mathrm{F}, \mathrm{Q}_1)$ of the crosscut Q_1 with respect to the continuum S^1. Since the computation is independent of the choice of the Junction [FMOT07], we can choose a junction J_v with junction point $v \in Q_1$ so that each of the three rays R_+, R_i and R_- intersect $F(Q_1)$ in exactly one point. Hence $\mathrm{Var}(\mathrm{F}, \mathrm{Q}_1) = +1$. Since φ is an orientation preserving homeomorphism, $\mathrm{Var}(\mathrm{f}, \varphi(\mathrm{Q}_1)) = +1$ and we are done.

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \square$

4. Applications

The results in the previous section can be used to obtain results in complex dynamics (see for example [BCO08]). We will show that in certain cases continua (e.g., impressions of external rays) must be degenerate. Suppose that $P : \mathbb{C} \to \mathbb{C}$ is a complex polynomial of degree d with a connected Julia set J. Let the filled-in Julia set be denoted by $K = T(J)$. We denote the external rays of K by R_α. It is well known [DH85a] that if the degree of P is d and $\sigma : \mathbb{C} \to \mathbb{C}$ is defined by $\sigma(z) = z^d$, then $P(R_\alpha) = R_{\sigma(\alpha)}$.

Let for $\lambda \in \mathbb{C}$, $L_\lambda : \mathbb{C} \to \mathbb{C}$ be defined by $L_\lambda(z) = \lambda z$. Suppose that p is a fixed point in J and $\lambda = f'(p)$ with $|\lambda| > 1$ (i.e., p is a *repelling* fixed point). Then there exists neighborhoods $U \subset V$ of p and a conformal isomorphism $\varphi : V \to \mathbb{D}$ such that for all $z \in U$, $P(z) = \varphi^{-1} \circ L_\lambda \circ \varphi(z)$. Now, a fixed point p is *parabolic* if $P'(p) = e^{2\pi i r}$ for some rational number $r \in \mathbb{Q}$. A nice description of the local dynamics at a parabolic fixed point can be found in [Mil00].

If p is a repelling or parabolic fixed point then [DH85a] there exist $k \geq 1$ external rays $R_{\alpha(i)}$ such that $\sigma|_{\{\alpha(1),\dots,\alpha(k)\}} : \{\alpha(1), \dots, \alpha(k)\} \to \{\alpha(1), \dots, \alpha(k)\}$ is a permutation, $P(R_{\alpha(i)}) = R_{\sigma(\alpha(i))}$, for each j, $R_{\alpha(j)}$ lands on p and no other external rays land on p. Also, if $P(R_{\alpha(i)}) = R_{\alpha(i)}$ for some i, then $\sigma(\alpha(j)) = \alpha(j)$ for all j. It is known that two distinct external rays are not homotopic in the complement of K. Given an external ray R_α of K we denote by $\Pi(\alpha) = \overline{R_\alpha} \setminus R_\alpha$ the *principle set of* α, and by $\mathrm{Imp}(\alpha)$ the *impression of* α (see [Mil00]). Given a set $A \subset S^1$, we extend the above notation by $\Pi(A) = \bigcup_{\alpha \in A} \Pi(\alpha)$ and $\mathrm{Imp}(A) = \bigcup_{\alpha \in A} \mathrm{Imp}(\alpha)$. Let $X \subset K$ be a non-separating continuum or a point such that:

(1) Pairwise disjoint non-separating continua/points $E_1 \subset X, \dots, E_m \subset X$ and finite sets of angles $A_1 = \{\alpha_1^1, \dots, \alpha_{i_1}^1\}, \dots, A_m = \{\alpha_1^m, \dots, \alpha_{i_m}^m\}$ are given with $i_k \geq 2, 1 \leq k \leq m$.

(2) We have $\Pi(A_j) \subset E_j$ (so the set $E_j \cup (\cup_{k=1}^{i_j} R_{\alpha_k^j}) = E_j'$ is closed and connected).

(3) X intersects a unique component C of $\mathbb{C} \setminus \cup E_j'$, such that $X \setminus \bigcup E_j = C \cap K$.

We call such X a *general puzzle-piece* and call the continua E_i the *exit continua* of X. Observe that if U is a Fatou domain then either a general puzzle-piece X contains U, or it is disjoint from U. For each j, the set E_j' divides the plane into i_j open sets which we will call *wedges (at E_j)*; denote by W_j the wedge which contains $X \setminus E_j$.

Let us now consider the condition (1) of Theorem 3.19. It is easy to see that applied "as is" to the polynomial P at parabolic points it is actually not true. Indeed, as explained above the local index at parabolic fixed points at which the derivative equals 1 is greater than 1. And indeed, in our case there are fixed rays landing at all fixed points, therefore [Mil00] the derivatives at all the parabolic points in X are equal to 1. The idea which allows us to solve this problem is that we can change our map P inside the parabolic domains in question without compromising the rest of the arguments and making these parabolic points topologically repelling. The thus constructed new map g will satisfy conditions of Theorem 3.19.

pararepel **Lemma 4.1.** *Suppose that X is a continuum and $p \in X$ is a parabolic point of a polynomial f and R is a fixed external which lands at p. Then f repels outside X at p.*

Proof. Let $p \in X$ be a parabolic fixed point and let F_i be the parabolic domains containing p in their boundaries B_i. Since there are fixed rays landing at p, all F_i's are forward invariant. By a nice recent result of Yin and Roesch [RY08], the boundary B_i of each F_i is a simple closed curve and $f|_{B_i}$ is conjugate to the map $z \to z^{d(i)}$ for some $d(i) \geq 2$. Let $\psi : F_i \to \mathbb{D}$ be a conformal isomorphism. Since $\mathrm{Bd}(F_i)$ is a simple closed curve, ψ extends to a homeomorphism. Since $f|_{B_i}$ is conjugate to the map $z \to z^{d(i)}$, it now follows that the map $P|_{\overline{F_i}}$ can be replaced by a map topologically conjugate by ψ to the map $g_i(z) = z^{d(i)}$ on the closed unit disk. Let g be the map defined by $g(z) = P(z)$ for each $z \in \mathbb{C} \setminus \bigcup F_i$ and $g(z) = g_i(z)$ when $z \in F_i$. Then g is clearly a positively oriented map.

The well-known analysis of the dynamics of P around parabolic points [Mil00] implies that P repels points away from p outside parabolic domains F_i. In other words, we can find a sequence of simple closed curves S_i which satisfy conditions of Definition 3.15 and show that p is a topologically repelling point of g. Hence the local index $\mathrm{Ind}(g, p)$ at p equals 1. On the other hand, by Lemma 3.21 and properties of X it follows that g repels outside X at p. Since f and g coincide outside X, f also repels at p. $\square$

The following corollary follows from Theorem 3.19.

pointdyn **Corollary 4.2.** *Suppose that $X \subset K$ is a non-separating continuum or a point. Then the following claims hold for X.*

1   (1) *Suppose that X is a general puzzle-piece with exit continua $E_1, \ldots, E_m$ such that either $P(E_i) \subset W_i$, or E_i is a fixed point. If X does not contain an invariant parabolic domain, all fixed points which belong to X are repelling or parabolic, and all rays landing at them are fixed, then X is a repelling or parabolic fixed point.*

2   (2) *Suppose that $X \subset J$ is an invariant continuum, all fixed points which belong to X are repelling or parabolic, and all rays landing at them are fixed. Then X is a repelling or parabolic fixed point.*

Proof. By way of contradiction we can assume that X is not a point. Let us show that no parabolic domain with a fixed point on its boundary can intersect X. Indeed, in the case (2) $X \subset J$ and no Fatou domain intersects X, so there is nothing to prove. In the case (1) observe that since X is a general puzzle-piece, it has to contain

the closure of the entire parabolic domain with a fixed point, say, p on its boundary. Then the fact that all external rays landing at p are fixed implies that all parabolic domains containing p in their boundaries are invariant. Since by the assumptions X contains no invariant parabolic domain, it does not contain any of them. So, X is disjoint from all parabolic domains containing a fixed point in their boundaries.

To apply Theorem 3.19 we need to verify that its conditions apply. It is easier to check the condition (2) first. To do so, observe first that $f(X) \cap X \neq \emptyset$. Indeed, otherwise no set E_i is a fixed point and $f(X)$ must be contained in one of the wedges formed by some E_l' but not in the wedge W_l. This implies that E_l neither is a fixed point, nor is mapped in W_l, a contradiction. Thus, $f(X) \cap X \neq \emptyset$ and we can think of $f(X)$ as a continuum which "grows" out of X. Now, any component of $f(X) \setminus X$ which intersects E_k for some k must be contained in one of the wedges at E_k, but not in W_k. Take the closure of their union and then its topological hull union E_i and denote it by Z_i. It is easy to check now that with these sets Z_i the map P scrambles the boundary of X. Moreover, if E_i is mapped into W_i then clearly $P(E_i) \cap Z_i = \emptyset$ (because Z_i is contained in the other wedges at E_i but is disjoint from W_i). On the other hand, if E_i is a fixed point then it is a repelling or parabolic fixed point with a few external fixed rays landing at it. Hence in a small neighborhood U_i of E_i the intersection $U_i \cap X$ maps into X as desired in the condition (2) of Theorem 3.19.

By Lemmas 3.21 and 4.1 P repels outside X at any fixed point in X. Moreover, using the map g constructed in the proof of Lemma 4.1, we see that g is topologically repelling at p and, hence $\mathrm{Ind}(g, p) = +1$. Hence the conditions of Theorem 3.19 are satisfied for the map g. Thus, by Theorem 3.19 we conclude that X is a point as desired. □

The following is an immediate corollary of Theorem 4.2.

`rot-neutr` **Corollary 4.3.** *Suppose that for a non-separating non-degenerate continuum $X \subset K$ one of the following facts hold.*

(1) *X is a general puzzle-piece with exit continua $E_1, \ldots, E_m$ such that either $P(E_i) \subset W_i$, or E_i is a fixed point.*
(2) *$X \subset J$ is an invariant continuum.*

Then either X contains a non-repelling and non-parabolic fixed point, or X contains an invariant parabolic domain, or X contains a repelling or parabolic fixed point at which a non-fixed ray lands.

Finally, the following corollary is useful in proving the degeneracy of certain impressions and establishing local continuity of the Julia set at some points.

Corollary 4.4. *Let $P : \mathbb{C} \to \mathbb{C}$ be a complex polynomial and R_α is a fixed external ray landing on a repelling or parabolic fixed point $p \in J$. Suppose that $T(Imp(\alpha))$ contains only repelling or parabolic periodic points. Then $Imp(\alpha)$ is degenerate.*

Proof. Let $X = \mathrm{Imp}(\alpha)$. Since R_α is a fixed external ray, $P(X) \subset X$. Clearly P does not rotate at p. Suppose that p' is another fixed point of P in X and R_β is an external ray landing at p'. Then $P(R_\beta)$ also lands on p'. If P rotates at p', then p' is a cut point of X. This would contradict the fact that $X = \mathrm{Imp}(\alpha)$. Hence P does not rotate at any fixed point in X and the result follows from Corollary 4.2. □

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